

FACTORIZATION SEMIGROUPS AND IRREDUCIBLE COMPONENTS OF HURWITZ SPACE. II

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ABSTRACT. This article is a continuation of the article with the same title (see arXiv:1003.2953v1). Let $\text{HUR}_{d,t}^G(\mathbb{P}^1)$ be the Hurwitz space of degree d coverings of the projective line \mathbb{P}^1 with Galois group G and having fixed monodromy type t consisting of a collection of local monodromy types (that is, a collection of conjugacy classes of permutations σ of the symmetric group \mathcal{S}_d acting on the set $I_d = \{1, \dots, d\}$). We prove that if the type t contains big enough number of local monodromies belonging to the conjugacy class C of an odd permutation σ which leaves fixed $f_C \geq 2$ elements of I_d , then the Hurwitz space $\text{HUR}_{d,t}^{\mathcal{S}_d}(\mathbb{P}^1)$ is irreducible.

INTRODUCTION

This article is a continuation of article [1]. To formulate the results of presented article, let us recall main definitions and notations used in [1]. A collection (S, G, α, ρ) , where S is a semigroup, G is a group, and $\alpha : S \rightarrow G$, $\rho : G \rightarrow \text{Aut}(S)$ are homomorphisms, is called a *semigroup S over a group G* if for all $s_1, s_2 \in S$ we have

$$s_1 \cdot s_2 = \rho(\alpha(s_1))(s_2) \cdot s_1 = s_2 \cdot \lambda(\alpha(s_2))(s_1), \quad (1)$$

where $\lambda(g) = \rho(g^{-1})$. Let $(S_1, G, \alpha_1, \rho_1)$ and $(S_2, G, \alpha_2, \rho_2)$ be two semigroups over a group G . A homomorphism of semigroups $\varphi : S_1 \rightarrow S_2$ is said *to be defined over G* if $\alpha_1(s) = \alpha_2(\varphi(s))$ and $\rho_2(g)(\varphi(s)) = \varphi(\rho_1(g)(s))$ for all $s \in S_1$ and $g \in G$.

A pair (G, O) , where $O \subset G$ is a subset of a group G invariant under the inner automorphisms, is called an *equipped group*. To each equipped group (G, O) one can associate a semigroup $S_O = S(G, O)$ over G (called the *factorization semigroup of the elements of G with factors in O*) generated by the elements of the alphabet $X = X_O = \{x_g \mid g \in O\}$ being subject to the following relations:

$$x_{g_1} \cdot x_{g_2} = x_{g_2} \cdot x_{g_2^{-1}g_1g_2} = x_{g_1g_2g_1^{-1}} \cdot x_{g_1} \quad (2)$$

for each $x_{g_1}, x_{g_2} \in X$ and if $g_2 = \mathbf{1}$ then $x_{g_1} \cdot x_{\mathbf{1}} = x_{g_1}$. The map $\alpha : X \rightarrow G$, given by $\alpha(x_g) = g$ for each $x_g \in X$, induces a homomorphism $\alpha : S_O \rightarrow G$ called the *product homomorphism*. The action (from the left) ρ of the group G on S_O is defined by the action on the alphabet X as follows:

$$x_a \in X \mapsto \rho(g)(x_a) = x_{gag^{-1}} \in X$$

for each $g \in G$. Note that $\alpha(\rho(g)(s)) = g\alpha(s)g^{-1}$ for all $s \in S_O$ and all $g \in G$.

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Let $O \setminus \{1\} = C_1 \sqcup \dots \sqcup C_m$ be the decomposition of the set O into the disjoint union of conjugacy classes of elements of G . Each element $s = x_{g_1} \cdot \dots \cdot x_{g_n} \in S_O$ defines an element $\tau(s) = n_1 C_1 + \dots + n_m C_m$ of the free abelian semigroup generated by C_1, \dots, C_m ($\tau(s)$ is called the *type* of s), where n_i is equal to the number of factors x_{g_j} entering in the factorization $s = x_{g_1} \cdot \dots \cdot x_{g_n}$ for which $g_j \in C_i$. The total number $n = \sum_{i=1}^m n_i$ is called the length of s and it is denoted by $ln(s)$. A subsemigroup S of S_G is called *stable* if there is an element $s \in S$ (called a *stabilizing element* of S) such that $s_1 \cdot s = s_2 \cdot s$ for any $s_1, s_2 \in S$ such that $\alpha(s_1) = \alpha(s_2)$ and $\tau(s_1) = \tau(s_2)$.

For an element $s = x_{g_1} \cdot \dots \cdot x_{g_n} \in S_O$ one can associate a subgroup $G_s = \langle g_1, \dots, g_n \rangle$ of G generated by the elements g_1, \dots, g_n . For each two (not necessary proper) subgroups H and Γ of G one can define subsemigroups $S_O^H = \{s \in S(G, O) \mid G_s = H\}$ and $S_{O, \Gamma} = \{s \in S(G, O) \mid \alpha(s) \in \Gamma\}$. If H and Γ are normal subgroups of G , then $S_{O, \Gamma}$ and S_O^H are semigroups over G . By definition, $S_{O, \Gamma}^H = S_{O, \Gamma} \cap S_O^H$.

Let \mathcal{S}_d be the symmetric group acting on the set $I_d = \{1, \dots, d\}$ and $T_d \subset \mathcal{S}_d$ be the subset of transpositions. The semigroup $S_{\mathcal{S}_d}$ is denoted by Σ_d . By Theorem 2.3. in [1], the element

$$h = \left(\prod_{i=1}^{d-1} x_{(i, i+1)} \right)^3$$

is a stabilizing element of Σ_d , where $(i, i+1) \in T_d$ is a transposition permuting the elements i and $i+1$ of I_d .

The aim of this article is to generalize this result to the case of almost all odd elements of the symmetric group \mathcal{S}_d . More precisely, let $C = C_\sigma$ be the conjugacy class of a permutation $\sigma \in \mathcal{S}_d$. As is known, if σ is an odd permutation, then the elements of C generate the group \mathcal{S}_d and, in particular, any transposition $(i, j) \in \mathcal{S}_d$ is a product of some permutations belonging to C . Denote by m_C the minimal number of permutations of C needed to express $(1, 2)$ as a product of permutations of C and fix one of the such expressions:

$$(1, 2) = \sigma_1 \dots \sigma_{m_C}, \quad \sigma_i \in C. \quad (3)$$

Denote by n_C the order of $\sigma \in C$, by $k_C = |C|$ the number of elements of C , and by f_C the number of elements of I_d fixed under the action of $\sigma \in C$ on I_d .

Theorem 1. *Let C be the conjugacy class of an odd permutation $\sigma \in \mathcal{S}_d$. If $f_C \geq 2$ then there is a constant*

$$N = N_C < 3^{d-3}(2d-1)(d-1)m_C + n_C k_C + 1$$

such that any element $s = \tilde{s} \cdot \bar{s} \in \Sigma_d^{S_d}$, where $\bar{s} \in S_C$, is uniquely defined by $\tau(s)$ and $\alpha(s)$ if $ln(\bar{s}) \geq N$.

Corollary 1. *Let an equipped symmetric group (\mathcal{S}_d, O) be such that the set O contains a conjugacy class C of an odd permutation σ , $f_C \geq 2$. Then $S_O = S(\mathcal{S}_d, O)$ is a stable semigroup.*

Note that in general case the constant N_C , existence of which is claimed in Theorem 1, is greater than 1. For example, as it was shown in [2], this is the case when C is the conjugacy class of the permutation $\sigma = (1, 2)(3, 4, 5) \in \mathcal{S}_8$.

The proof of Theorem 1 is similar to the proof of Theorem 2.3 in [1] and it is based on the following theorem.

Theorem 2. *Let C be the conjugacy class of an odd permutation $\sigma \in \mathcal{S}_d$ and an element $\bar{s}_{(i_1, i_2)} \in S_C$ be such that*

- (i) $\alpha(\bar{s}_{(i_1, i_2)}) = (i_1, i_2)$,
- (ii) *there are $i_3, i_4 \in I_d \setminus \{i_1, i_2\}$ such that $\rho((i_3, i_4))(\bar{s}_{(i_1, i_2)}) = \bar{s}_{(i_1, i_2)}$.*

Then there is an embedding over \mathcal{S}_d of the semigroup S_{T_d} in S_C .

Let $\text{HUR}_{d,b}(\mathbb{P}^1)$ (resp., $\text{HUR}_{d,b}^G(\mathbb{P}^1)$) be the Hurwitz space of ramified degree d coverings of the projective line \mathbb{P}^1 (defined over \mathbb{C}) branched over b points (and resp., with Galois group G). In [1], it was shown that the irreducible components of $\text{HUR}_{d,b}(\mathbb{P}^1)$ are in one to one correspondence with the orbits of the action of \mathcal{S}_d by simultaneous conjugations on $\Sigma_{d,1,b} = \{s \in \Sigma_{d,1} \mid \text{ln}(s) = b\}$ (that is, of the action defined by the homomorphism ρ) and if $G = \mathcal{S}_d$, then the irreducible components of $\text{HUR}_{d,b}^{\mathcal{S}_d}(\mathbb{P}^1)$ are in one to one correspondence with the elements of $\Sigma_{d,1}^{\mathcal{S}_d}$ of length equal to b . If an irreducible component of $\text{HUR}_{d,b}^{\mathcal{S}_d}(\mathbb{P}^1)$ corresponds to an element $s \in \Sigma_{d,1}^{\mathcal{S}_d}$, then we call $\tau(s)$ *monodromy factorization type* of the coverings belonging to this component. Denote by $\text{HUR}_{d,t}^{\mathcal{S}_d}(\mathbb{P}^1)$ the union of irreducible components corresponding to the elements $s \in \Sigma_{d,1}^{\mathcal{S}_d}$ with $\tau(s) = t$.

As a corollary of Theorem 1 we obtain

Theorem 3. *Let C be the conjugacy class of an odd permutation $\sigma \in \mathcal{S}_d$ such that $f_C \geq 2$. If the monodromy factorization type t contains more than N_C factors belonging to C , where N_C is defined in Theorem 1, then the space $\text{HUR}_{d,t}^{\mathcal{S}_d}(\mathbb{P}^1)$ is irreducible.*

1. PROOF OF THEOREM 2

Without loss of generality, we can assume that $(i_1, i_2) = (1, 2)$ and $(i_3, i_4) = (3, 4)$.

For each $(i, j) \in T_d$ let us choose an element $\sigma_{i,j} \in \mathcal{S}_d$ such that $(i, j) = \sigma_{i,j}(1, 2)\sigma_{i,j}^{-1}$ and put

$$c = \bar{s}_{(1,2)}^2 \cdot \bar{s}_{(2,3)}^2 \cdot \dots \cdot \bar{s}_{(d-1,d)}^2,$$

where $\bar{s}_{(i,j)} = \rho(\sigma_{i,j})(\bar{s}_{(1,2)})$.

Obviously, we have $\alpha(\bar{s}_{(i,j)}) = (i, j)$ and $\alpha(c) = \mathbf{1}$. Since the transpositions $(1, 2), \dots, (d-1, d)$ generate the group \mathcal{S}_d , then $c \in S_{C,1}^{\mathcal{S}_d}$. Therefore, by Proposition 1.1 (2) in [1], the element c is fixed under the conjugation action of \mathcal{S}_d on S_C .

For $k \geq 4$ let us denote by $Z_k \simeq \mathcal{S}_2 \times \mathcal{S}_{k-2}$ a subgroup of \mathcal{S}_d generated by transpositions $(1, 2)$ and (i, j) , $3 \leq i < j \leq k$. Note that Z_d is the centralizer in \mathcal{S}_d of the transposition $(1, 2)$.

Claim 1. *There is an element $z_{(1,2)} \in S_C$ such that $\alpha(z_{(1,2)}) = (1, 2)$ and $\rho(\sigma)(z_{(1,2)}) = z_{(1,2)}$ for each $\sigma \in Z_d$*

Proof. By induction in k , let us show that there is an element $y_{(1,2),k} \in S_C^{\mathcal{S}_d}$ such that $\alpha(y_{(1,2),k}) = (1, 2)$ and $\rho(\sigma)(y_{(1,2),k}) = y_{(1,2),k}$ for each $\sigma \in Z_k$. Then $z_{(1,2)} = y_{(1,2),d}$ is a desired element.

Put $y_{(1,2),4} = \bar{s}_{(1,2)} \cdot c$. If we move the first factor $\bar{s}_{(1,2)}$ to the right, then we obtain

$$\begin{aligned} y_{(1,2),4} &= \bar{s}_{(1,2)} \cdot \bar{s}_{(1,2)} \cdot \bar{s}_{(1,2)} \cdot \bar{s}_{(2,3)}^2 \cdot \dots \cdot \bar{s}_{(d-1,d)}^2 = \\ &= \rho((1, 2))(\bar{s}_{(1,2)}) \cdot \bar{s}_{(1,2)} \cdot \bar{s}_{(1,2)} \cdot \bar{s}_{(2,3)}^2 \cdot \dots \cdot \bar{s}_{(d-1,d)}^2 = \\ &= \rho((1, 2))(\bar{s}_{(1,2)}) \cdot c = \rho((1, 2))(\bar{s}_{(1,2)}) \cdot \rho((1, 2))(c) = \\ &= \rho((1, 2))(\bar{s}_{(1,2)} \cdot c) = \rho((1, 2))(y_{(1,2),4}), \end{aligned}$$

since c is fixed under the conjugation action of \mathcal{S}_d .

Similarly, by assumption of Theorem 2,

$$\rho((3, 4))(y_{(1,2),4}) = \rho((3, 4))(\bar{s}_{(1,2)} \cdot c) = \rho((3, 4))(\bar{s}_{(1,2)}) \cdot \rho((3, 4))(c) = \bar{s}_{(1,2)} \cdot c = y_{(1,2),4}$$

and hence $\rho(\sigma)(y_{(1,2),4}) = y_{(1,2),4}$ for all $\sigma \in Z_4$.

Assume that for some $k \geq 4$, $k < d$, we constructed an element $y_{(1,2),k} \in S_C^{\mathcal{S}_d}$ such that $\alpha(y_{(1,2),k}) = (1, 2)$ and $\rho(\sigma)(y_{(1,2),k}) = y_{(1,2),k}$ for all $\sigma \in Z_k$. Consider an element $y'_{(1,2),k} = \rho((k, k+1))(y_{(1,2),k})$. Obviously, $y'_{(1,2),k}$ belongs to $S_C^{\mathcal{S}_d}$ and it is easy to see that $\alpha(y'_{(1,2),k}) = (1, 2)$. Hence, the element $y_{(1,2),k} \cdot y'_{(1,2),k}$ belongs to $S_{C,1}^{\mathcal{S}_d}$ and therefore it is fixed under the conjugation action of \mathcal{S}_d . Besides, the element $y'_{(1,2),k}$ is fixed under the action of the group Z'_k generated by transpositions $(i, j) \in Z_{k+1}$, $i, j \neq k$. Indeed, if $(i, j) \in Z'_k$ and $i, j \neq k+1$, then

$$\begin{aligned} \rho((i, j))(y'_{(1,2),k}) &= \rho((i, j))(\rho((k, k+1))(y_{(1,2),k})) = \\ &= \rho((i, j)(k, k+1))(y_{(1,2),k}) = \rho((k, k+1)(i, j))(y_{(1,2),k}) = \\ &= \rho((k, k+1))(\rho((i, j))(y_{(1,2),k})) = \rho((k, k+1))(y_{(1,2),k}) = y'_{(1,2),k}. \end{aligned}$$

If $(i, k+1) \in Z'_k$, then

$$\begin{aligned} \rho((i, k+1))(y'_{(1,2),k}) &= \rho((i, k+1))(\rho((k, k+1))(y_{(1,2),k})) = \\ &= \rho((i, k+1)(k, k+1))(y_{(1,2),k}) = \rho((k, k+1)(i, k))(y_{(1,2),k}) = \\ &= \rho((k, k+1))(\rho((i, k))(y_{(1,2),k})) = \rho((k, k+1))(y_{(1,2),k}) = y'_{(1,2),k}, \end{aligned}$$

since $(i, k) \in Z_k$.

Moreover, the elements $y_{(1,2),k}$, $y'_{(1,2),k}$ commute. Indeed,

$$\begin{aligned} y'_{(1,2),k} \cdot y_{(1,2),k} &= \rho(\alpha(y'_{(1,2),k}))(y_{(1,2),k}) \cdot y'_{(1,2),k} = \\ &= \rho((1, 2))(y_{(1,2),k}) \cdot y'_{(1,2),k} = y_{(1,2),k} \cdot y'_{(1,2),k}. \end{aligned}$$

Put $y_{(1,2),k+1} := y_{(1,2),k} \cdot y'_{(1,2),k}$. It is easy to see that $y_{(1,2),k+1} \in S_C^{\mathcal{S}_d}$ and $\alpha(y_{(1,2),k+1}) = (1, 2)$. Let us show that $\rho(\sigma)(y_{(1,2),k+1}) = y_{(1,2),k+1}$ for each $\sigma \in Z_{k+1}$. First of all note that the group Z_{k+1} is generated by the elements of the groups Z_k and Z'_k .

For each $\sigma \in Z_k$ we have

$$\begin{aligned}\rho(\sigma)(y_{(1,2),k+1}) &= \rho(\sigma)(y_{(1,2),k} \cdot y_{(1,2),k} \cdot y'_{(1,2),k}) = \\ \rho(\sigma)(y_{(1,2),k}) \cdot \rho(\sigma)(y_{(1,2),k} \cdot y'_{(1,2),k}) &= y_{(1,2),k} \cdot y_{(1,2),k} \cdot y'_{(1,2),k},\end{aligned}$$

since the element $y_{(1,2),k} \cdot y'_{(1,2),k} \in S_{C,1}^{S_d}$ is fixed under the conjugation action of \mathcal{S}_d .

Similarly, for each $\sigma \in Z'_k$ we have

$$\begin{aligned}\rho(\sigma)(y_{(1,2),k+1}) &= \rho(\sigma)(y_{(1,2),k}^2 \cdot y'_{(1,2),k}) = \\ \rho(\sigma)(y_{(1,2),k}^2) \cdot \rho(\sigma)(y'_{(1,2),k}) &= y_{(1,2),k}^2 \cdot y'_{(1,2),k} = y_{(1,2),k+1},\end{aligned}$$

since the element $y_{(1,2),k} \cdot y_{(1,2),k} \in S_{C,1}^{S_d}$ is fixed under the conjugation action of \mathcal{S}_d . Claim 1 is proved. \square

Consider the orbit $X_{T_{C,d}}$ of the element $z_{(1,2)}$ under the conjugation action of \mathcal{S}_d on the semigroup S_C , where $z_{(1,2)}$ is the element constructed in the proof of Claim 1 with the help of the element $\bar{s}_{(1,2)}$.

Claim 2. *The map $\bar{\alpha} : X_{T_{C,d}} \rightarrow X_{T_d} = \{x_{(i,j)} \mid (i,j) \in T_d\}$ given by $\bar{\alpha}(\rho(\sigma)(z_{(1,2)})) = x_{\sigma(1,2)\sigma^{-1}}$ is one-to-one correspondence.*

Proof. The map $\bar{\alpha} : X_{T_{C,d}} \rightarrow X_{T_d}$ is surjective, since for each $(i,j) \in T_d$ there is $\sigma \in \mathcal{S}_d$ such that $(i,j) = \sigma(1,2)\sigma^{-1}$ and for this σ we have

$$\alpha(\rho(\sigma)(z_{(1,2)})) = \sigma(1,2)\sigma^{-1} = (i,j),$$

$$\alpha(\bar{\alpha}(\rho(\sigma)(z_{(1,2)}))) = \alpha(x_{\sigma(1,2)\sigma^{-1}}) = \sigma(1,2)\sigma^{-1} = (i,j).$$

The order of the group Z_d is equal to $2(d-2)!$. Therefore, by Claim 1, the number $|X_{T_{C,d}}|$ of the elements of $X_{T_{C,d}}$ is not more than $\frac{d!}{2(d-2)!} = \frac{d(d-1)}{2} = |T_d|$ and hence $\bar{\alpha} : X_{T_{C,d}} \rightarrow X_{T_d}$ is one-to-one. \square

Denote by $z_{(i,j)}$ an element $z \in X_{T_{C,d}}$ such that $\alpha(z) = (i,j)$ and by $S_{T_{C,d}}$ a subsemigroup of S_C generated by the elements $z_{(i,j)}$, $1 \leq i, j \leq d$, $i \neq j$.

Claim 3. *The subsemigroup $S_{T_{C,d}}$ of S_C is a semigroup over \mathcal{S}_d whose generators $z_{(i,j)}$, $1 \leq i, j \leq d$, $i \neq j$, are subjected to the relations*

$$\begin{aligned}z_{(i,j)} &= z_{(j,i)} \text{ for all } \{i,j\}_{ord} \subset I_d; \\ z_{(i_1,i_2)} \cdot z_{(i_1,i_3)} &= z_{(i_2,i_3)} \cdot z_{(i_1,i_2)} = z_{(i_1,i_3)} \cdot z_{(i_2,i_3)} \text{ for all } \{i_1,i_2,i_3\}_{ord} \subset I_d; \\ z_{(i_1,i_2)} \cdot z_{(i_3,i_4)} &= z_{(i_3,i_4)} \cdot z_{(i_1,i_2)} \text{ for all } \{i_1,i_2,i_3,i_4\}_{ord} \subset I_d.\end{aligned} \tag{4}$$

Proof. Evident. \square

For $s \in S_{T_{C,d}}$, where s is a product of n generators $z_{(i,j)}$ of $S_{T_{C,d}}$, we define the T -length of s as $ln_T(s) = n$. Denote by $\bar{\alpha}^{-1}$ the map inverse to $\bar{\alpha}$. By Claim 3, the map $\bar{\alpha}^{-1} : X_{T_d} \rightarrow X_{T_{C,d}}$ can be extended to a homomorphism $\bar{\alpha}^{-1} : S_{T_d} \rightarrow$

$S_{T_{C,d}}$ of semigroups over \mathcal{S}_d . Note that $\ln(s) = \ln_T(\bar{\alpha}^{-1}(s))$ for $s \in S_{T_d}$. Define a subsemigroup $S_{T_{C,d}}^{\mathcal{S}_d, T}$ of $S_{T_{C,d}}$ as follows: $S_{T_{C,d}}^{\mathcal{S}_d, T} = \bar{\alpha}^{-1}(S_{T_d}^{\mathcal{S}_d})$.

Now, Theorem 2 follows from

Claim 4. *The homomorphism $\bar{\alpha}^{-1} : S_{T_d} \rightarrow S_{T_{C,d}}$ of semigroups over \mathcal{S}_d is an isomorphism and, consequently, all statements of subsections 2.2, 2.3, and Lemma 2.9 in [1] remain true if we substitute $x_{(i,j)} \in S_{T_d}$ by $z_{(i,j)} \in S_{T_{C,d}}$, change the length of elements by T -length, and change $S_{T_d}^{\mathcal{S}_d}$ by $S_{T_{C,d}}^{\mathcal{S}_d, T}$.*

Proof. Evident. □

2. PROOF OF THEOREM 1

Let us consider an element $\bar{s}_{(1,2)} = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_{m_C}}$, where $\sigma_1, \dots, \sigma_{m_C} \in C$ are the factors in factorization (3).

If $f_C \geq 2$ then we can and will assume that all σ_i entering into factorization (3) belong to the subgroup $\mathcal{S}_d^{\{3,4\}} \simeq \mathcal{S}_{d-2}$ of \mathcal{S}_d the elements of which leave fixed the elements $3, 4 \in I_d$. Therefore the element $\bar{s}_{(1,2)} = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_{m_C}}$ satisfies all conditions of Theorem 2 and, consequently, the elements $z_{(i,j)}$, constructed in section 1 with the help of $\bar{s}_{(1,2)} = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_{m_C}}$, generate in S_C a semisubgroup isomorphic to S_{T_d} over the group \mathcal{S}_d .

Note that the length of the element $z_{(1,2)}$, constructed in the proof of Claim 1, is equal to $\ln(z_{(1,2)}) = 3^{d-4}(2d-1)m_C$ if we start from the element $\bar{s}_{(1,2)} = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_{m_C}}$, where $\sigma_1, \dots, \sigma_{m_C}$ are the factors of expression (3).

Denote by

$$h_C = (z_{(1,2)} \cdot z_{(2,3)} \cdot \dots \cdot z_{(d-1,d)})^3.$$

Rewrite h_C as a product

$$h_C = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_L}, \quad \sigma_i \in C \text{ for } i = 1, \dots, L.$$

It is easy to see that

$$\ln(h_C) = 3^{d-3}(2d-1)(d-1)m_C := L.$$

To prove Theorem 1, we need the following

Claim 5. *Under the conditions of Theorem 1, let an element $s = \tilde{s} \cdot \bar{s} \in \Sigma_d^{\mathcal{S}_d}$ be such that $\bar{s} \in S_C$ of length $\ln(\bar{s}) := M \geq 3^{d-3}(2d-1)(d-1)m_C + n_C k_C$. Then the element s can be represented as $s = \tilde{s}' \cdot h_C$.*

Proof. Let

$$\bar{s} = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_M}, \tag{5}$$

$\sigma_i \in C$. Since $M = \ln(\bar{s}) \geq 3^{d-3}(2d-1)(d-1)m_C + n_C k_C > n_C k_C$, then for some $\sigma \in C$ there are at least $n_C + 1$ factors entering into factorization (5) equal to σ .

Therefore \bar{s} can be written in the form: $\bar{s} = \bar{s}' \cdot x_\sigma^{n_C}$, where $\bar{s}' \in S_C$ is such that $\tilde{s} \cdot \bar{s}' \in \Sigma_d^{S_d}$. By Lemma 1.1 in [1], we have

$$s = \tilde{s} \cdot \bar{s}' \cdot x_\sigma^{n_C} = \tilde{s} \cdot \bar{s}' \cdot x_{\sigma_L}^{n_C} = \tilde{s} \cdot \bar{s}_L \cdot x_{\sigma_L},$$

where $\bar{s}_L = \bar{s}' \cdot x_{\sigma_L}^{n_C-1}$. Note that $\tilde{s} \cdot \bar{s}_L \in \Sigma_d^{S_d}$ and $ln(\bar{s}_L) > n_C k_C$. Therefore, by the same arguments, the element $\tilde{s} \cdot \bar{s}_L$ can be written in the form: $\tilde{s} \cdot \bar{s}_L = \tilde{s} \cdot \bar{s}'_L \cdot x_{\sigma_{L-1}}^{n_C-1} \cdot x_{\sigma_{L-1}}$. Put $\bar{s}_{L-1} = \bar{s}'_L \cdot x_{\sigma_{L-1}}^{n_C-1}$. Repeating the same arguments for $\tilde{s} \cdot \bar{s}_{L-1}$ we obtain that $\tilde{s} \cdot \bar{s}_{L-1} = \tilde{s} \cdot \bar{s}_{L-2} \cdot x_{\sigma_{L-1}}$, and so on. Finally, on the L th step we obtain that

$$s = \tilde{s} \cdot \bar{s} = \tilde{s} \cdot \bar{s}_0 \cdot (x_{\sigma_1} \cdot \dots \cdot x_{\sigma_L}) = \tilde{s} \cdot \bar{s}_0 \cdot h_C. \quad \square$$

Now to complete the proof of Theorem 1, recall that the proof of Theorem 3.2 in [1] consists of two parts. In the first part of the proof, for any element $s = \tilde{s} \cdot \bar{s} \in \Sigma_d^{S_d}$, where $\bar{s} \in S_{T_d}$ has the length $ln(\bar{s}) \geq 3(d-1)$, it was proved the existence of another factorization $s = \tilde{s}_1 \cdot \bar{s}_1$ such that $\bar{s}_1 \in S_{T_d}^{S_d}$ with $ln(\bar{s}_1) = 3(d-1)$. In this case the element \bar{s}_1 is uniquely determined by its product $\alpha(\bar{s}_1) = \alpha(\tilde{s}_1)^{-1} \alpha(s)$. In the second part of the proof of Theorem 3.2 in [1] it was proved that for a such factorization $s = \tilde{s}_1 \cdot \bar{s}_1$ there is another factorization $s = \tilde{s}_2 \cdot \bar{s}_2$, where again $\bar{s}_2 \in S_{T_d}^{S_d}$ has the length $ln(\bar{s}_2) = 3(d-1)$ and \tilde{s}_2 is uniquely determined by the type $\tau(\tilde{s}_1)$. The proof of the last statement used only properties of the semigroup S_{T_d} and relations (1) in the factorization semigroups. Therefore, by Claims 4 and 5, the end of the proof of Theorem 1 coincides with the second part of the proof of Theorem 3.2 in [1]. \square

REFERENCES

- [1] Vik.S. Kulikov: *Factorization semigroups and irreducible components of Hurwitz space*. arXiv:1003.2953v1 (to appear in Izv. Math.).
- [2] B. Wajnryb: *Orbits of Hurwitz action for coverings of a sphere with two special fibres*. Indag. Math. (N.S.), vol. 7 (1996), no. 4, 549 – 558.

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FACTORIZATION SEMIGROUPS AND IRREDUCIBLE COMPONENTS OF HURWITZ SPACE. II

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ABSTRACT. This article is a continuation of the article [1]. Let $\text{HUR}_{d,t}^{\mathcal{S}_d}(\mathbb{P}^1)$ be the Hurwitz space of degree d coverings of the projective line \mathbb{P}^1 with Galois group \mathcal{S}_d and having fixed monodromy type t consisting of a collection of local monodromy types (that is, a collection of conjugacy classes of permutations σ of the symmetric group \mathcal{S}_d acting on the set $I_d = \{1, \dots, d\}$). We prove that if the type t contains big enough number of local monodromies belonging to the conjugacy class C of an odd permutation σ which leaves fixed $f_C \geq 2$ elements of I_d , then the Hurwitz space $\text{HUR}_{d,t}^{\mathcal{S}_d}(\mathbb{P}^1)$ is irreducible.

INTRODUCTION

This article is a continuation of article [1]. To formulate the results of presented article, let us recall main definitions and notations used in [1]. A collection (S, G, α, ρ) , where S is a semigroup, G is a group, and $\alpha : S \rightarrow G$, $\rho : G \rightarrow \text{Aut}(S)$ are homomorphisms, is called a *semigroup S over a group G* if for all $s_1, s_2 \in S$ we have

$$s_1 \cdot s_2 = \rho(\alpha(s_1))(s_2) \cdot s_1 = s_2 \cdot \lambda(\alpha(s_2))(s_1), \quad (1)$$

where $\lambda(g) = \rho(g^{-1})$. Let $(S_1, G, \alpha_1, \rho_1)$ and $(S_2, G, \alpha_2, \rho_2)$ be two semigroups over a group G . A homomorphism of semigroups $\varphi : S_1 \rightarrow S_2$ is said to be *defined over G* if $\alpha_1(s) = \alpha_2(\varphi(s))$ and $\rho_2(g)(\varphi(s)) = \varphi(\rho_1(g)(s))$ for all $s \in S_1$ and $g \in G$.

A pair (G, O) , where $O \subset G$ is a subset of a group G invariant under the inner automorphisms, is called an *equipped group*. To each equipped group (G, O) one can associate a semigroup $S_O = S(G, O)$ over G (called the *factorization semigroup of the elements of G with factors in O*) generated by the elements of the alphabet $X = X_O = \{x_g \mid g \in O\}$ being subject to the following relations:

$$x_{g_1} \cdot x_{g_2} = x_{g_2} \cdot x_{g_2^{-1}g_1g_2} = x_{g_1g_2g_1^{-1}} \cdot x_{g_1} \quad (2)$$

for each $x_{g_1}, x_{g_2} \in X$ and if $g_2 = \mathbf{1}$ then $x_{g_1} \cdot x_{\mathbf{1}} = x_{g_1}$. The map $\alpha : X \rightarrow G$, given by $\alpha(x_g) = g$ for each $x_g \in X$, induces a homomorphism $\alpha : S_O \rightarrow G$ called the *product homomorphism*. The action (from the left) ρ of the group G on S_O is defined by the action on the alphabet X as follows:

$$x_a \in X \mapsto \rho(g)(x_a) = x_{gag^{-1}} \in X$$

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for each $g \in G$. Note that $\alpha(\rho(g)(s)) = g\alpha(s)g^{-1}$ for all $s \in S_O$ and all $g \in G$.

Let $O \setminus \{1\} = C_1 \sqcup \dots \sqcup C_m$ be the decomposition of the set O into the disjoint union of conjugacy classes of elements of G . Each element $s = x_{g_1} \cdot \dots \cdot x_{g_n} \in S_O$ defines an element $\tau(s) = n_1 C_1 + \dots + n_m C_m$ of the free abelian semigroup generated by C_1, \dots, C_m ($\tau(s)$ is called the *type* of s), where n_i is equal to the number of factors x_{g_j} entering in the factorization $s = x_{g_1} \cdot \dots \cdot x_{g_n}$ for which $g_j \in C_i$. The total number $n = \sum_{i=1}^m n_i$ is called the length of s and it is denoted by $ln(s)$. A subsemigroup S of S_G is called *stable* if there is an element $s \in S$ (called a *stabilizing element* of S) such that $s_1 \cdot s = s_2 \cdot s$ for any $s_1, s_2 \in S$ such that $\alpha(s_1) = \alpha(s_2)$ and $\tau(s_1) = \tau(s_2)$.

For an element $s = x_{g_1} \cdot \dots \cdot x_{g_n} \in S_O$ one can associate a subgroup $G_s = \langle g_1, \dots, g_n \rangle$ of G generated by the elements g_1, \dots, g_n . For each two (not necessary proper) subgroups H and Γ of G one can define subsemigroups $S_O^H = \{s \in S(G, O) \mid G_s = H\}$ and $S_{O,\Gamma} = \{s \in S(G, O) \mid \alpha(s) \in \Gamma\}$. If H and Γ are normal subgroups of G , then $S_{O,\Gamma}$ and S_O^H are semigroups over G . By definition, $S_{O,\Gamma}^H = S_{O,\Gamma} \cap S_O^H$.

Let \mathcal{S}_d be the symmetric group acting on the set $I_d = \{1, \dots, d\}$ and $T_d \subset \mathcal{S}_d$ be the subset of transpositions. The semigroup $S_{\mathcal{S}_d}$ is denoted by Σ_d . By Theorem 2.3. in [1], the element

$$h = \left(\prod_{i=1}^{d-1} x_{(i,i+1)} \right)^3$$

is a stabilizing element of Σ_d , where $(i, i+1) \in T_d$ is a transposition permuting the elements i and $i+1$ of I_d .

The aim of this article is to generalize this result to the case of almost all odd elements of the symmetric group \mathcal{S}_d . More precisely, let $C = C_\sigma$ be the conjugacy class of a permutation $\sigma \in \mathcal{S}_d$. Denote by n_C the order of $\sigma \in C$, by $k_C = |C|$ the number of elements of C , and by f_C the number of elements of I_d fixed under the action of $\sigma \in C$ on I_d .

As is known, if σ is an odd permutation, then the elements of C generate the group \mathcal{S}_d and, in particular, any transposition $(i, j) \in \mathcal{S}_d$ is a product of some permutations belonging to C . In the case $f_C \geq 2$, denote by m_C the minimal number (counted with multiplicities) of permutations of $C \cap \mathcal{S}_{d-2}$ needed to express $(1, 2)$ as a product of permutations of C and fix one of the such expressions:

$$(1, 2) = \sigma_1 \dots \sigma_{m_C}, \quad \sigma_i \in C \cap \mathcal{S}_{d-2}. \quad (3)$$

Theorem 1. *Let C be the conjugacy class of an odd permutation $\sigma \in \mathcal{S}_d$. If $f_C \geq 2$ then there is a constant*

$$N = N_C < 3^{d-3}(2d-1)(d-1)m_C + n_C k_C + 1$$

such that any element $s = \tilde{s} \cdot \bar{s} \in \Sigma_d^{\mathcal{S}_d}$, where $\bar{s} \in S_C$, is uniquely defined by $\tau(s)$ and $\alpha(s)$ if $ln(\bar{s}) \geq N$.

Corollary 1. *Let an equipped symmetric group (\mathcal{S}_d, O) be such that the set O contains a conjugacy class C of an odd permutation σ , $f_C \geq 2$. Then $S_O = S(\mathcal{S}_d, O)$ is a stable semigroup.*

Note that in general case the constant N_C , existence of which is claimed in Theorem 1, is greater than 1. For example, as it was shown in [2], this is the case when C is the conjugacy class of the permutation $\sigma = (1, 2)(3, 4, 5) \in \mathcal{S}_8$.

The proof of Theorem 1 is similar to the proof of Theorem 2.3 in [1] and it is based on the following theorem.

Theorem 2. *Let C be the conjugacy class of an odd permutation $\sigma \in \mathcal{S}_d$ and an element $\bar{s}_{(i_1, i_2)} \in S_C$ be such that*

- (i) $\alpha(\bar{s}_{(i_1, i_2)}) = (i_1, i_2)$,
- (ii) *there are $i_3, i_4 \in I_d \setminus \{i_1, i_2\}$ such that $\rho((i_3, i_4))(\bar{s}_{(i_1, i_2)}) = \bar{s}_{(i_1, i_2)}$.*

Then there is an embedding over \mathcal{S}_d of the semigroup $S_{T_d}^{\mathcal{S}_d}$ in S_C .

Let $\text{HUR}_{d,b}(\mathbb{P}^1)$ (resp., $\text{HUR}_{d,b}^G(\mathbb{P}^1)$) be the Hurwitz space of ramified degree d coverings of the projective line \mathbb{P}^1 (defined over \mathbb{C}) branched over b points (and resp., with Galois group G). In [1], it was shown that the irreducible components of $\text{HUR}_{d,b}(\mathbb{P}^1)$ are in one to one correspondence with the orbits of the action of \mathcal{S}_d by simultaneous conjugations on $\Sigma_{d,1,b} = \{s \in \Sigma_{d,1} \mid \text{ln}(s) = b\}$ (that is, of the action defined by the homomorphism ρ) and if $G = \mathcal{S}_d$, then the irreducible components of $\text{HUR}_{d,b}^{\mathcal{S}_d}(\mathbb{P}^1)$ are in one to one correspondence with the elements of $\Sigma_{d,1}^{\mathcal{S}_d}$ of length equal to b . If an irreducible component of $\text{HUR}_{d,b}^{\mathcal{S}_d}(\mathbb{P}^1)$ corresponds to an element $s \in \Sigma_{d,1}^{\mathcal{S}_d}$, then we call $\tau(s)$ *monodromy factorization type* of the coverings belonging to this component. Denote by $\text{HUR}_{d,t}^{\mathcal{S}_d}(\mathbb{P}^1)$ the union of irreducible components corresponding to the elements $s \in \Sigma_{d,1}^{\mathcal{S}_d}$ with $\tau(s) = t$.

As a corollary of Theorem 1 we obtain

Theorem 3. *Let C be the conjugacy class of an odd permutation $\sigma \in \mathcal{S}_d$ such that $f_C \geq 2$. If the monodromy factorization type t contains more than N_C factors belonging to C , where N_C is defined in Theorem 1, then the space $\text{HUR}_{d,t}^{\mathcal{S}_d}(\mathbb{P}^1)$ is irreducible.*

Note that for Hurwitz spaces of d -sheeted coverings of the disc $\Delta = \{z \in \mathbb{C} \mid |z| \leq 1\}$ (respectively, of d -sheeted coverings of the affine line \mathbb{C}^1), a statement, similar to Theorem 3, is also true.

1. PROOF OF THEOREM 2

Without loss of generality, we can assume that $(i_1, i_2) = (1, 2)$ and $(i_3, i_4) = (3, 4)$.

For each $(i, j) \in T_d$ let us choose an element $\sigma_{i,j} \in \mathcal{S}_d$ such that $(i, j) = \sigma_{i,j}(1, 2)\sigma_{i,j}^{-1}$ and put

$$c = \bar{s}_{(1,2)}^2 \cdot \bar{s}_{(2,3)}^2 \cdot \dots \cdot \bar{s}_{(d-1,d)}^2,$$

where $\bar{s}_{(i,j)} = \rho(\sigma_{i,j})(\bar{s}_{(1,2)})$.

Obviously, we have $\alpha(\bar{s}_{(i,j)}) = (i, j)$ and $\alpha(c) = \mathbf{1}$. Since the transpositions $(1, 2), \dots, (d-1, d)$ generate the group \mathcal{S}_d , then $c \in S_{C,1}^{\mathcal{S}_d}$. Therefore, by Proposition 1.1 (2) in [1], the element c is fixed under the conjugation action of \mathcal{S}_d on S_C .

For $k \geq 4$ let us denote by $Z_k \simeq \mathcal{S}_2 \times \mathcal{S}_{k-2}$ a subgroup of \mathcal{S}_d generated by transpositions $(1, 2)$ and (i, j) , $3 \leq i < j \leq k$. Note that Z_d is the centralizer in \mathcal{S}_d of the transposition $(1, 2)$.

Claim 1. *There is an element $z_{(1,2)} \in S_C$ such that $\alpha(z_{(1,2)}) = (1, 2)$ and $\rho(\sigma)(z_{(1,2)}) = z_{(1,2)}$ for each $\sigma \in Z_d$*

Proof. By induction in k , let us show that there is an element $y_{(1,2),k} \in S_C^{\mathcal{S}_d}$ such that $\alpha(y_{(1,2),k}) = (1, 2)$ and $\rho(\sigma)(y_{(1,2),k}) = y_{(1,2),k}$ for each $\sigma \in Z_k$. Then $z_{(1,2)} = y_{(1,2),d}$ is a desired element.

Put $y_{(1,2),4} = \bar{s}_{(1,2)} \cdot c$. If we move the first factor $\bar{s}_{(1,2)}$ to the right, then we obtain

$$\begin{aligned} y_{(1,2),4} &= \bar{s}_{(1,2)} \cdot \bar{s}_{(1,2)} \cdot \bar{s}_{(1,2)} \cdot \bar{s}_{(2,3)}^2 \cdot \dots \cdot \bar{s}_{(d-1,d)}^2 = \\ &= \rho((1, 2))(\bar{s}_{(1,2)}) \cdot \bar{s}_{(1,2)} \cdot \bar{s}_{(1,2)} \cdot \bar{s}_{(2,3)}^2 \cdot \dots \cdot \bar{s}_{(d-1,d)}^2 = \\ &= \rho((1, 2))(\bar{s}_{(1,2)}) \cdot c = \rho((1, 2))(\bar{s}_{(1,2)}) \cdot \rho((1, 2))(c) = \\ &= \rho((1, 2))(\bar{s}_{(1,2)} \cdot c) = \rho((1, 2))(y_{(1,2),4}), \end{aligned}$$

since c is fixed under the conjugation action of \mathcal{S}_d .

Similarly, by assumption of Theorem 2,

$$\rho((3, 4))(y_{(1,2),4}) = \rho((3, 4))(\bar{s}_{(1,2)} \cdot c) = \rho((3, 4))(\bar{s}_{(1,2)}) \cdot \rho((3, 4))(c) = \bar{s}_{(1,2)} \cdot c = y_{(1,2),4}$$

and hence $\rho(\sigma)(y_{(1,2),4}) = y_{(1,2),4}$ for all $\sigma \in Z_4$.

Assume that for some $k \geq 4$, $k < d$, we constructed an element $y_{(1,2),k} \in S_C^{\mathcal{S}_d}$ such that $\alpha(y_{(1,2),k}) = (1, 2)$ and $\rho(\sigma)(y_{(1,2),k}) = y_{(1,2),k}$ for all $\sigma \in Z_k$. Consider an element $y'_{(1,2),k} = \rho((k, k+1))(y_{(1,2),k})$. Obviously, $y'_{(1,2),k}$ belongs to $S_C^{\mathcal{S}_d}$ and it is easy to see that $\alpha(y'_{(1,2),k}) = (1, 2)$. Hence, the element $y_{(1,2),k} \cdot y'_{(1,2),k}$ belongs to $S_{C,1}^{\mathcal{S}_d}$ and therefore it is fixed under the conjugation action of \mathcal{S}_d . Besides, the element $y'_{(1,2),k}$ is fixed under the action of the group Z'_k generated by transpositions $(i, j) \in Z_{k+1}$, $i, j \neq k$. Indeed, if $(i, j) \in Z'_k$ and $i, j \neq k+1$, then

$$\begin{aligned} \rho((i, j))(y'_{(1,2),k}) &= \rho((i, j))(\rho((k, k+1))(y_{(1,2),k})) = \\ &= \rho((i, j)(k, k+1))(y_{(1,2),k}) = \rho((k, k+1)(i, j))(y_{(1,2),k}) = \\ &= \rho((k, k+1))(\rho((i, j))(y_{(1,2),k})) = \rho((k, k+1))(y_{(1,2),k}) = y'_{(1,2),k}. \end{aligned}$$

If $(i, k+1) \in Z'_k$, then

$$\begin{aligned} \rho((i, k+1))(y'_{(1,2),k}) &= \rho((i, k+1))(\rho((k, k+1))(y_{(1,2),k})) = \\ &= \rho((i, k+1)(k, k+1))(y_{(1,2),k}) = \rho((k, k+1)(i, k))(y_{(1,2),k}) = \\ &= \rho((k, k+1))(\rho((i, k))(y_{(1,2),k})) = \rho((k, k+1))(y_{(1,2),k}) = y'_{(1,2),k}, \end{aligned}$$

since $(i, k) \in Z_k$.

Moreover, the elements $y_{(1,2),k}, y'_{(1,2),k}$ commute. Indeed,

$$\begin{aligned} y'_{(1,2),k} \cdot y_{(1,2),k} &= \rho(\alpha(y'_{(1,2),k}))(y_{(1,2),k}) \cdot y'_{(1,2),k} = \\ \rho((1,2))(y_{(1,2),k}) \cdot y'_{(1,2),k} &= y_{(1,2),k} \cdot y'_{(1,2),k}. \end{aligned}$$

Put $y_{(1,2),k+1} := y_{(1,2),k}^2 \cdot y'_{(1,2),k}$. It is easy to see that $y_{(1,2),k+1} \in S_C^{\mathcal{S}_d}$ and $\alpha(y_{(1,2),k+1}) = (1,2)$. Let us show that $\rho(\sigma)(y_{(1,2),k+1}) = y_{(1,2),k+1}$ for each $\sigma \in Z_{k+1}$. First of all note that the group Z_{k+1} is generated by the elements of the groups Z_k and Z'_k .

For each $\sigma \in Z_k$ we have

$$\begin{aligned} \rho(\sigma)(y_{(1,2),k+1}) &= \rho(\sigma)(y_{(1,2),k} \cdot y_{(1,2),k} \cdot y'_{(1,2),k}) = \\ \rho(\sigma)(y_{(1,2),k}) \cdot \rho(\sigma)(y_{(1,2),k} \cdot y'_{(1,2),k}) &= y_{(1,2),k} \cdot y_{(1,2),k} \cdot y'_{(1,2),k}, \end{aligned}$$

since the element $y_{(1,2),k} \cdot y'_{(1,2),k} \in S_{C,1}^{\mathcal{S}_d}$ is fixed under the conjugation action of \mathcal{S}_d .

Similarly, for each $\sigma \in Z'_k$ we have

$$\begin{aligned} \rho(\sigma)(y_{(1,2),k+1}) &= \rho(\sigma)(y_{(1,2),k}^2 \cdot y'_{(1,2),k}) = \\ \rho(\sigma)(y_{(1,2),k}^2) \cdot \rho(\sigma)(y'_{(1,2),k}) &= y_{(1,2),k}^2 \cdot y'_{(1,2),k} = y_{(1,2),k+1}, \end{aligned}$$

since the element $y_{(1,2),k} \cdot y_{(1,2),k} \in S_{C,1}^{\mathcal{S}_d}$ is fixed under the conjugation action of \mathcal{S}_d . Claim 1 is proved. \square

Consider the orbit $X_{T_{C,d}}$ of the element $z_{(1,2)}$ under the conjugation action of \mathcal{S}_d on the semigroup S_C , where $z_{(1,2)}$ is the element constructed in the proof of Claim 1 with the help of the element $\bar{s}_{(1,2)}$.

Claim 2. *The map $\bar{\alpha} : X_{T_{C,d}} \rightarrow X_{T_d} = \{x_{(i,j)} \mid (i,j) \in T_d\}$ given by $\bar{\alpha}(\rho(\sigma)(z_{(1,2)})) = x_{\sigma(1,2)\sigma^{-1}}$ is one-to-one correspondence.*

Proof. The map $\bar{\alpha} : X_{T_{C,d}} \rightarrow X_{T_d}$ is surjective, since for each $(i,j) \in T_d$ there is $\sigma \in \mathcal{S}_d$ such that $(i,j) = \sigma(1,2)\sigma^{-1}$ and for this σ we have

$$\alpha(\rho(\sigma)(z_{(1,2)})) = \sigma(1,2)\sigma^{-1} = (i,j),$$

$$\alpha(\bar{\alpha}(\rho(\sigma)(z_{(1,2)}))) = \alpha(x_{\sigma(1,2)\sigma^{-1}}) = \sigma(1,2)\sigma^{-1} = (i,j).$$

The order of the group Z_d is equal to $2(d-2)!$. Therefore, by Claim 1, the number $|X_{T_{C,d}}|$ of the elements of $X_{T_{C,d}}$ is not more than $\frac{d!}{2(d-2)!} = \frac{d(d-1)}{2} = |T_d|$ and hence $\bar{\alpha} : X_{T_{C,d}} \rightarrow X_{T_d}$ is one-to-one. \square

Denote by $z_{(i,j)}$ an element $z \in X_{T_{C,d}}$ such that $\alpha(z) = (i,j)$ and by $S_{T_{C,d}}$ a subsemigroup of S_C generated by the elements $z_{(i,j)}$, $1 \leq i, j \leq d$, $i \neq j$.

Claim 3. *The subsemigroup $S_{T_{C,d}}$ of S_C is a semigroup over \mathcal{S}_d . The elements $z_{(i,j)} \in S_{T_{C,d}}$, $1 \leq i, j \leq d$, $i \neq j$, satisfy the following relations*

$$\begin{aligned} z_{(i,j)} &= z_{(j,i)} \text{ for all } \{i,j\}_{ord} \subset I_d; \\ z_{(i_1,i_2)} \cdot z_{(i_1,i_3)} &= z_{(i_2,i_3)} \cdot z_{(i_1,i_2)} = z_{(i_1,i_3)} \cdot z_{(i_2,i_3)} \text{ for all } \{i_1,i_2,i_3\}_{ord} \subset I_d; \\ z_{(i_1,i_2)} \cdot z_{(i_3,i_4)} &= z_{(i_3,i_4)} \cdot z_{(i_1,i_2)} \text{ for all } \{i_1,i_2,i_3,i_4\}_{ord} \subset I_d. \end{aligned} \tag{4}$$

Proof. It directly follows from constructions of the elements $z_{(i,j)}$ and Claim 1.1 in [1]. \square

Claim 4. *The map $\bar{\alpha}^{-1} : X_{T_d} \rightarrow X_{T_{C,d}}$ is extended to surjective homomorphism $\bar{\alpha}^{-1} : S_{T_d} \rightarrow S_{T_{C,d}}$ of semigroups over group \mathcal{S}_d .*

Proof. Note that if to substitute $x_{(i,j)}$ instead of $z_{(i,j)}$ in relations (4), then we obtain the defining relations in the semigroup S_{T_d} . Therefore, it follows from Claim 3 that $\bar{\alpha}^{-1}$ can be extended to a surjective homomorphism semigroups over group \mathcal{S}_d . \square

For $s \in S_{T_{C,d}}$, where s is a product of n generators $z_{(i,j)}$ of $S_{T_{C,d}}$, we define the T -length of s as $ln_T(s) = n$. We have $ln(s) = ln_T(\bar{\alpha}^{-1}(s))$ for $s \in S_{T_d}$.

Note that it follows from Claim 4 that any statement in [1], in which it is claimed that an element of S_{T_d} can be represented as the product of some generators $x_{i,j}$, is true for elements of $S_{T_{C,d}}$ if in the statement we change the elements $x_{(i,j)}$ by $z_{(i,j)}$ and change the lengths by T -lengths.

Define a subsemigroup $S_{T_{C,d}}^{\mathcal{S}_d, T}$ of $S_{T_{C,d}}$ as follows: $S_{T_{C,d}}^{\mathcal{S}_d, T} = \bar{\alpha}^{-1}(S_{T_d}^{\mathcal{S}_d})$.

Theorem 2 follows from

Claim 5. *The restriction of $\bar{\alpha}^{-1} : S_{T_d} \rightarrow S_{T_{C,d}}$ to $S_{T_d}^{\mathcal{S}_d}$,*

$$\bar{\alpha}^{-1} : S_{T_d}^{\mathcal{S}_d} \rightarrow S_{T_{C,d}}^{\mathcal{S}_d, T},$$

is an isomorphism of semigroups over the group \mathcal{S}_d .

Proof. It follows from Theorem 2.1 in [1] that the homomorphism $\bar{\alpha}^{-1} : S_{T_d}^{\mathcal{S}_d} \rightarrow S_{T_{C,d}}^{\mathcal{S}_d, T}$ is injective. \square

Note also that Theorem 2.1 in [1] and Claim 5 imply directly the following

Corollary 2. *The elements s of the semigroup $S_{T_{C,d}}^{\mathcal{S}_d, T}$ are defined uniquely by $\alpha(s)$ and $ln_T(s)$.*

2. PROOF OF THEOREM 1

Let us consider an element $\bar{s}_{(1,2)} = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_{m_C}}$, where $\sigma_1, \dots, \sigma_{m_C} \in C$ are the factors in factorization (3).

If $f_C \geq 2$ then we can and will assume that all σ_i entering into factorization (3) belong to the subgroup $\mathcal{S}_d^{\{3,4\}} \simeq \mathcal{S}_{d-2}$ of \mathcal{S}_d the elements of which leave fixed the elements $3, 4 \in I_d$. Therefore the element $\bar{s}_{(1,2)} = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_{m_C}}$ satisfies all conditions of Theorem 2 and, consequently, the elements $z_{(i,j)}$, constructed in section 1 with the help of $\bar{s}_{(1,2)} = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_{m_C}}$, define uniquely a semisubgroup $S_{T_{C,d}}^{\mathcal{S}_d, T}$ of S_C isomorphic to S_{T_d} over the group \mathcal{S}_d .

Note that the length of the element $z_{(1,2)}$, constructed in the proof of Claim 1, is equal to $ln(z_{(1,2)}) = 3^{d-4}(2d-1)m_C$ if we start from the element $\bar{s}_{(1,2)} = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_{m_C}}$, where $\sigma_1, \dots, \sigma_{m_C}$ are the factors of expression (3).

Denote by

$$h_C = (z_{(1,2)} \cdot z_{(2,3)} \cdot \dots \cdot z_{(d-1,d)})^3.$$

Rewrite h_C as a product

$$h_C = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_L}, \quad \sigma_i \in C \text{ for } i = 1, \dots, L.$$

It is easy to see that

$$\ln(h_C) = 3^{d-3}(2d-1)(d-1)m_C := L.$$

To prove Theorem 1, we need the following

Claim 6. *Under the conditions of Theorem 1, let an element $s = \tilde{s} \cdot \bar{s} \in \Sigma_d^{S_d}$ be such that $\bar{s} \in S_C$ of length $\ln(\bar{s}) := M \geq 3^{d-3}(2d-1)(d-1)m_C + n_C k_C$. Then the element s can be represented as $s = \tilde{s}' \cdot h_C$.*

Proof. Let

$$\bar{s} = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_M}, \tag{5}$$

$\sigma_i \in C$. Since $M = \ln(\bar{s}) \geq 3^{d-3}(2d-1)(d-1)m_C + n_C k_C > n_C k_C$, then for some $\sigma \in C$ there are at least $n_C + 1$ factors entering into factorization (5) equal to σ . Therefore \bar{s} can be written in the form: $\bar{s} = \bar{s}' \cdot x_{\sigma}^{n_C}$, where $\bar{s}' \in S_C$ is such that $\tilde{s} \cdot \bar{s}' \in \Sigma_d^{S_d}$. By Lemma 1.1 in [1], we have

$$s = \tilde{s} \cdot \bar{s}' \cdot x_{\sigma}^{n_C} = \tilde{s} \cdot \bar{s}' \cdot x_{\sigma_L}^{n_C} = \tilde{s} \cdot \bar{s}_L \cdot x_{\sigma_L},$$

where $\bar{s}_L = \bar{s}' \cdot x_{\sigma_L}^{n_C-1}$. Note that $\tilde{s} \cdot \bar{s}_L \in \Sigma_d^{S_d}$ and $\ln(\bar{s}_L) > n_C k_C$. Therefore, by the same arguments, the element $\tilde{s} \cdot \bar{s}_L$ can be written in the form: $\tilde{s} \cdot \bar{s}_L = \tilde{s} \cdot \bar{s}'_L \cdot x_{\sigma_{L-1}}^{n_C-1} \cdot x_{\sigma_{L-1}}$. Put $\bar{s}_{L-1} = \bar{s}'_L \cdot x_{\sigma_{L-1}}^{n_C-1}$. Repeating the same arguments for $\tilde{s} \cdot \bar{s}_{L-1}$ we obtain that $\tilde{s} \cdot \bar{s}_{L-1} = \tilde{s} \cdot \bar{s}_{L-2} \cdot x_{\sigma_{L-2}}$, and so on. Finally, on the L th step we obtain that

$$s = \tilde{s} \cdot \bar{s} = \tilde{s} \cdot \bar{s}_0 \cdot (x_{\sigma_1} \cdot \dots \cdot x_{\sigma_L}) = \tilde{s} \cdot \bar{s}_0 \cdot h_C. \quad \square$$

Now to complete the proof of Theorem 1, recall that the proof of Theorem 3.2 in [1] consists of two parts. In the first part of the proof, for any element $s = \tilde{s} \cdot \bar{s} \in \Sigma_d^{S_d}$, where $\bar{s} \in S_{T_d}$ has the length $\ln(\bar{s}) \geq 3(d-1)$, it was proved the existence of another factorization $s = \tilde{s}_1 \cdot \bar{s}_1$ such that $\bar{s}_1 \in S_{T_d}^{S_d}$ with $\ln(\bar{s}_1) = 3(d-1)$. In this case the element \bar{s}_1 is uniquely determined by its product $\alpha(\bar{s}_1) = \alpha(\tilde{s}_1)^{-1}\alpha(s)$.

In the second part of the proof of Theorem 3.2 in [1] it was proved that for a such factorization $s = \tilde{s}_1 \cdot \bar{s}_1$ there is another factorization $s = \tilde{s}_2 \cdot \bar{s}_2$, where again $\bar{s}_2 \in S_{T_d}^{S_d}$ has the length $\ln(\bar{s}_2) = 3(d-1)$ and \bar{s}_2 is uniquely determined by the type $\tau(\tilde{s}_1)$. The proof of the last statement used only properties of the semigroup S_{T_d} and relations (1) in the factorization semigroups. Therefore, by Claims 5 and 6, the end of the proof of Theorem 1 coincides with the second part of the proof of Theorem 2.3 in [1]. Only, we must do the following changes: the elements $x_{(i,j)}$ are changed by $z_{(i,j)}$, the lengths of elements are changed by T -lengths, the element $h_{d,g}$ is changed by $\bar{\alpha}^{-1}(h_{d,g})$, the semigroup $S_{T_d}^{S_d}$ is changed by $S_{T_{C,d}}^{S_d, T}$, and the homomorphism r is changed by $\bar{\alpha}^{-1} \circ r$. \square

However, according the request of the referee, this proof is given once again. For this purpose, denote by $h_{C,d,g} = \bar{\alpha}^{-1}(h_{d,g})$ the image of the Hurwitz element $h_{d,g} = x_{(1,2)}^{2g} \cdot x_{(1,2)}^2 \cdot \dots \cdot x_{(d-1,d)}^2$.

Lemma 1. *For any disjoint union $\{i_{1,1}, \dots, i_{k_1,1}\} \sqcup \dots \sqcup \{i_{1,n}, \dots, i_{k_n,n}\}$ of ordered subsets of I_d the Hurwitz element $h_{C,d,0}$ can be represented as a product*

$$h_{C,d,0} = (z_{(i_{1,1}, i_{2,1})} \cdot \dots \cdot z_{(i_{k_1-1,1}, i_{k_1,1})}) \cdot \dots \cdot (z_{(i_{1,n}, i_{2,n})} \cdot \dots \cdot z_{(i_{k_n-1,n}, i_{k_n,n})}) \cdot \bar{h},$$

where \bar{h} is an element of $S_{T_{C,d}}^{S_d, T}$.

Proof. It directly follows from Lemma 2.9 in [1] and Claim 5. \square

By Claim 6, the element s can be represented as a product: $s = \tilde{s}' \cdot \bar{s}$, where \bar{s} is an element of $S_{T_{C,d}}^{S_d, T}$ of T -length $k \geq 3(d-1)$ (in our case $\bar{s} = h_C$ and $k = 3(d-1)$), and let $\tilde{s}' = x_{\sigma'_1} \cdot \dots \cdot x_{\sigma'_m}$. By Proposition 2.4 in [1] and Claim 5, we have $\bar{s} = h_{C,d,0} \cdot \bar{s}'$.

To complete the proof of Theorem 1, let us use induction on m . If $m = 0$ (that is, if $s \in S_{T_{C,d}}$), then Theorem 1 follows from Proposition 2.4 in [1] and Claim 5.

Let $m = 1$. For the canonical representative $\sigma_{m,0}$ of type $t(\sigma_m)$ (the definition of the canonical representative is given in [1]), there is an element $\bar{\sigma}_m \in \mathcal{S}_d$ such that $\sigma_{m,0} = \bar{\sigma}_m^{-1} \sigma'_m \bar{\sigma}_m$. The permutation $\bar{\sigma}_m$ can be factorized into the product of cyclic permutations and each cyclic permutation can be factorized into the product of transpositions:

$$\bar{\sigma}_m = ((i_{1,1}, i_{2,1}) \dots (i_{k_1-1,1}, i_{k_1,1})) \dots ((i_{1,n}, i_{2,n}) \dots (i_{k_n-1,n}, i_{k_n,n})).$$

Consider the element

$$\bar{r}(x_{\bar{\sigma}_m}) = (z_{(i_{1,1}, i_{2,1})} \cdot \dots \cdot z_{(i_{k_1-1,1}, i_{k_1,1})}) \cdot \dots \cdot (z_{(i_{1,n}, i_{2,n})} \cdot \dots \cdot z_{(i_{k_n-1,n}, i_{k_n,n})}) \in S_{T_{C,d}}.$$

By Lemma 1, we have

$$h_{C,d,0} = \bar{r}(x_{\bar{\sigma}_m}) \cdot \bar{h}_m,$$

where \bar{h}_m is an element of $S_{T_{C,d}}^{S_d, T}$. We have

$$\begin{aligned} s &= x_{\sigma'_m} \cdot h_{d,0} \cdot \bar{s}' = x_{\sigma'_m} \cdot \bar{r}(x_{\bar{\sigma}_m}) \cdot \bar{h}_m \cdot \bar{s}' = \\ &\bar{r}(x_{\bar{\sigma}_m}) \cdot x_{\sigma_{m,0}} \cdot \bar{h}_m \cdot \bar{s}' = x_{\sigma_{m,0}} \cdot \bar{r}(x_{\bar{\sigma}'_m}) \cdot \bar{h}_m \cdot \bar{s}', \end{aligned}$$

where $x_{\bar{\sigma}'_m} = \lambda(\sigma_{m,0})(x_{\bar{\sigma}_m})$. We have $\bar{s}'_1 = \bar{r}(x_{\bar{\sigma}'_m}) \cdot \bar{h}_m \cdot \bar{s}' \in S_{T_{C,d}}^{S_d, T}$ and $\alpha(\bar{s}'_1) = \sigma_{m,0}^{-1} \alpha(s)$. Theorem 2.4 in [1] and Claim 5 imply that $\bar{s}'_1 = \bar{r}(x_\sigma) \cdot h_{C,d,g}$, where $\sigma = \alpha(\bar{s}'_1) = \sigma_{m,0}^{-1} \alpha(s)$ and $g = \frac{k - \ln_t(x_\sigma)}{2} - d + 1$.

Now assume that Theorem 1 is proved for all $m < m_0$ and consider an element

$$s = x_{\sigma_1} \cdot \dots \cdot x_{\sigma_{m_0}} \cdot \bar{s}_1,$$

where the element $\bar{s}_1 \in S_{T_{C,d}}^{S_d, T}$ has T -length equal $k \geq 3(d-1)$. We have

$$\begin{aligned} s &= x_{\sigma_1} \cdot \dots \cdot x_{\sigma_{m_0}} \cdot \bar{s}_1 = x_{\sigma'_2} \cdot \dots \cdot x_{\sigma'_{m_0}} \cdot x_{\sigma_1} \cdot \bar{s}_1 = \\ &x_{\sigma'_2} \cdot \dots \cdot x_{\sigma'_{m_0}} \cdot x_{\sigma_{1,0}} \cdot \bar{s}'_1 = x_{\sigma_{1,0}} \cdot x_{\sigma'_2} \cdot \dots \cdot x_{\sigma'_{m_0}} \cdot \bar{s}'_1, \end{aligned}$$

where $\sigma'_j = \sigma_1 \sigma_j \sigma_1^{-1}$ and $\sigma''_j = \sigma_{1,0}^{-1} \sigma'_j \sigma_{1,0}$ for $j = 2, \dots, m$, and the element $\bar{s}'_1 \in S_{T,d}^{S_d,T}$ has T -length $ln_T(\bar{s}'_1) = k$. It follows from inductive assumption that

$$s = x_{\sigma_{1,0}} \cdot (x_{\sigma'_2} \cdot \dots \cdot x_{\sigma''_{m_0}} \cdot \bar{s}'_1) = x_{\sigma_{1,0}} \cdot (x_{\sigma_{2,0}} \cdot \dots \cdot x_{\sigma_{m_0,0}} \cdot \bar{s}''_1),$$

where the element $\bar{s}''_1 \in S_{T,d}^{S_d,T}$ has T -length $ln_T(\bar{s}''_1) = k$. By Proposition 2.4 in [1] and Claim 5, we have $\bar{s}''_1 = \bar{r}(x_\sigma) \cdot h_{C,d,g}$, where $\sigma = \alpha(\bar{s}''_1) = (\sigma_{1,0} \dots \sigma_{m,0})^{-1} \alpha(s)$ and $g = \frac{k - ln_t(x_\sigma)}{2} - d + 1$. \square

REFERENCES

- [1] Vik.S. Kulikov: *Factorization semigroups and irreducible components of Hurwitz space*. Izv. Math., 75:4 (2011), 711748
- [2] B. Wajnryb: *Orbits of Hurwitz action for coverings of a sphere with two special fibres*. Indag. Math. (N.S.), vol. 7 (1996), no. 4, 549 – 558.

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